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ABSTRACT

Function Probe is a multi-representational software for Apple Macintosh computers. It was designed to allow students to approach problems in different ways and/or use different representations. This case study describes a 16-year-old student as he creates a path among a variety of representations of transformations of functions while using the Function Probe computer software. The model used in this study inverts the order of the traditional model of teaching about transformation of functions, by starting with visualization of graphs, then focusing on relationships between graphs and tabular values, and finally, focusing on relationships between graphs and algebraic representation. Contains 14 references. (MKR)

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The Role of the Teaching Experiment: Students' Construction of Transformations in a Multiple Representational Environment[1]

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1. Assumptions

In this paper, we will introduce a model based on visualization for the teaching and learning of transformations of functions. Before we get into the presentation of the model and the design of the study, we would like to describe to describe five assumptions that guided the design of the research:

a) Case studies can be fundamental to a research program that aims to improve mathematics education. Although there is a great difference between the classroom environment and the one of an interview, interviewing students allows us to take an in-depth look at students' reasoning while allowing students to spend the time they need to cope with a given problem. Often these two features are not possible in the classroom;

b) The over-dominance of algebra in the teaching of mathematics needs to be challenged. We would like to argue that models based on visualization, need careful consideration and equal curricular attention, especially now that graphical dynamical software is available;

c) An alternative to overemphasizing algebra lies in the use of multi-representational software such as Function Probe© (Confrey, 1991). The software design was a result of years of teaching and research done by the Mathematics Education Research Group at Cornell University[1]. The design of the software is therefore heavily influenced by students' reasoning. We see this as an ongoing process: the design of Function Probe (FP) was based on research on students' reasoning, and new research continues to inform the redesign of the software;

d) In a multi-representational tool, students can be provided with tasks that allow them to conduct their own investigations and engage in inquiry. Inquiry in this setting consists of a series of conjectures and coordination and contrasts of the use of representations;

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[1] This group is directed by Dr. Jere Confrey who is also the designer of FP together with Erick Smith (researcher) and Forrest Carrol (leading programer).

e) The particular design of a given software exerts significant impact on the constructions made by a student. As the student proceeds there is an interaction between the students' experience and prior constructions and the tool which can be described as a process of mutual "intershaping" (Borba, 1993).

Within the framework set by these assumptions, we will discuss the strategies used by **Ron**[2] as he was solving problems that provoked him to think about the relationship among transformations in graphs of a function, changes in tabular values of the function which was being transformed and changes in the coefficients of the equation of the function which was being transformed. However, before we do this we will introduce some features of the design of the study which include the model used for the learning of transformations of functions and some features of FP.

2. Notes on the Design of the Study and on Function Probe [3]

2.1 Procedure

To build models about how students think about transformations of functions, "teaching experiments" (Cobb & Steffe, 1983) were conducted. In this methodology, which are a variation on clinical interviews, students and researchers are expected to learn from each other, as the student proceeds through a series of tasks designed by the interviewer. In this methodology "the role of the interaction between interviewer and interviewee is acknowledged as an intervention, hence the label, teaching experiment. It is further assumed that, "when students engage with a genuine problematic, struggle to resolve it, and reflect on those actions, learning of a very profound sort has occurred." (Confrey, 1991b, p. 128). Ron was interviewed during approximately eight two-hour sessions over a five week time period. He

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[2] Ron was a 16 years old at that time that the interview took place. He was considered a good student by his teachers. He had completed Course 3, which included an introduction to functions but did not include transformations of functions as a topic. He was not taking any course at the time the interview started. Halfway through the teaching experiment, he started an independent study with a teacher from his school: The Alternative Community School in Ithaca, NY.

[3] A more thorough discussion of the design can be found in Borba (1993). To know about some of the strategies of Doug, another student who took part in the same study, see Borba & Confrey (1992) A complete discussion of Function Probe can be found in (Confrey et al., 1991).

had paper and pencil and an Apple Macintosh Computer (IISi) running FP available for his use.

2.2 A Description of the Software

FP is a multi-representational software for Apple Macintosh computers. It was designed to allow students to approach problems in different ways and/or using different representations. As the name of the software suggests, FP is designed to be used in solving problems which involve functional relationships, although problems involving other areas of mathematics might also be solved appropriately using this tool. At the time the teaching experiment was undertaken, the software had three windows: graph, table and calculator window.

The graph window allows the user to graph functions in a number of different ways: 1) using the analytical expression of the function; 2) plotting point by point in the (x,y) format; 3) sketching using the mouse as if it were a pen; 4) calling a "calculator button"; or 5) "receiving" ordered pairs from a table. In addition, FP has special

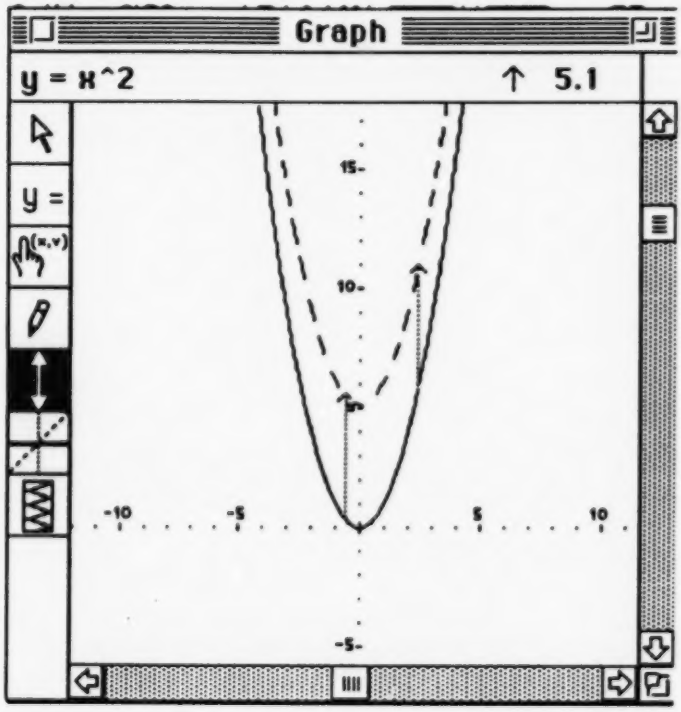

Figure 1 - A vertical translation by 5.1 made through direct actions on the graph. Notice that, since the command 'hide transformations' is on, the equation of the transformed parabola will not show on the left upper corner. The transformation could have been done through the keyboard. In this case the amount entered of the translation would show on the right upper corner.

features for transformations of functions. The graph of a function can be transformed by mouse actions using the bottom three icons on the left hand side of figure 1. The value for the transformation can also be typed directly in the upper right hand corner of the graph window if accuracy and scale requires it (see figure 1). In this sense, the icon palette in the graph window of Function Probe allows for actions - such as horizontal and vertical translations, horizontal and vertical stretches and reflections across vertical and horizontal lines - to be made "directly" on the graph. The algebraic results of the "mouse driven" transformations can be either hidden or displayed using a menu option. Scaling in the graph window is

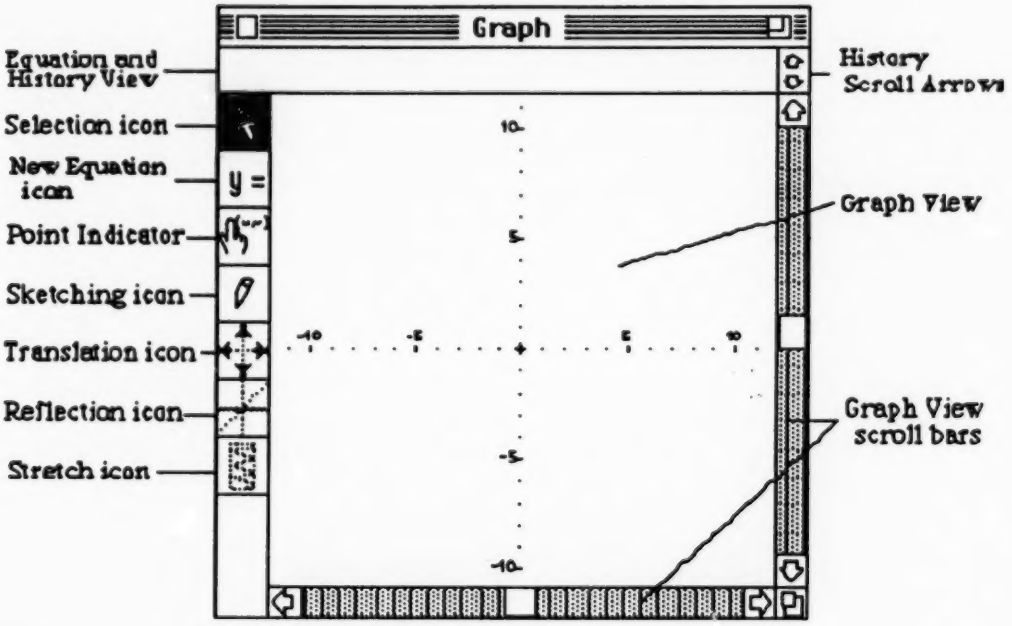

Figure 2 - An overview of the graph window.

not automatic. Re-scaling in the graph window requires the use of the "rescale box", so that students have to actively identify the intervals in the "x" and "y" axis that

they want to be displayed as well as the size of the unit they wish to be marked off in tics. Scales can be saved for reuse. A history of all equations drawn, sketched, deleted or transformed is kept in the history window.

In addition to the translations, stretch, reflection and new equation icons, the icon palette also has the sketch and the point indicator icon which are relevant for this study. The former allows students to make a free hand drawing of a graph, while the latter allows the identification of the coordinates of a point in the Cartesian plane. The sketch icon became very important as a vehicle for students' predictions and lent itself very well to interviewer probing. The point indicator can be used as a measuring tool. Another feature of FP, which was fundamental for this study was the sampling tool which is accessible through the "graph menu". The sampling tool allows a finite subset of the function to be selected and "sent" to the table[4]. In addition, "sampling" can be a powerful tool in the sense that it allows students in relating transformations in the graph with numerical changes in the ordered pairs of a function. This was used in the study to assist points to be selected and highlighted and/or sent to a table from a continuous graph to generate a subset of the function with discrete domain and range.

The graph window also has algebra facilities. After a graph is transformed, FP shows one possible equation for the transformed graph. FP has a feature that, when activated, hides the equation. Therefore, transformations can be done without showing the equation. For instance, on figure 1, the equation of the highlighted graph is not displayed after the graph is translated, instead "y=. . ." is shown.

The table window facilitates entry of data in tables, a task that would be otherwise tedious. "x" and "y" columns can be generated independently using the "fill" command. For instance, this command allows the user to fill one column in an arithmetic progression (e.g.. the "x" column starting at zero, adding one until 50) and another column in a geometric progression (e.g. the "y" column in a geometric progression starting at 100 with rate "1/2" until it reaches the fiftieth place). If one column has already been generated, a second column can be generated by an

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[4]The idea of adding the sampling tool to FP came as a result of the development of the design of study in which Ron was one of the students interviewed. This episode illustrates the interplay between research and software development.

equation that uses the variable of the first column as independent variable. In spite of how a pair of columns is generated, they can be "sent" to the graph window in the (x,y) form, where the function, having a discrete domain and range, is plotted in the Cartesian plane.

The calculator window, in addition to the regular features of a calculator, has other educational features. It keeps a history of the keys pushed by students to track students' work. It has many different buttons for functions such as exponential, quadratic and trigonometric functions. Buttons can also be created, allowing students to "program" the calculator to do an operation which they intend to use repeatedly. Such buttons can also be sent to the table - where they can be used to fill columns as the analytical expression of a function - and to the graph window. For instance, a column x can "store" a subset of the natural numbers such as [1,30] while the image can be stored in the "y" column and be described by the button $j_1 = (? - 5)*7 +3$, where j_n is the "n^{th}" button created and the "?" will play the role of a variable in a function (algebraically this is coded as y=(x-5)7). The data stored in this table can also be sent to the graph window.

The flexible, dynamic, and multi-representational aspect of Function Probe creates a unique environment for working with transformations of functions. In the study of which Ron took part, vertical and horizontal translations, reflections around vertical and horizontal lines, and vertical and horizontal stretches were studied. We will describe one way of describing the above listed transformations. We will start by discussing transformations on the graph and how these transformations can be "translated" to a table and an equation representation.[5]

2.3 The Model for the Teaching of Transformations

Traditionally transformations of functions have been taught with a strong emphasis on algebra. A typical approach is the one in which coefficients of a function would be changed and a question regarding changes in the graph would be asked. For instance, a typical problem on this topic could be: if $y=x^2+5$ has changed to $y=2x^2+5$, what did the graph of the transformed function look like? In the more recent

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[5] We do not claim to make "the" description of transformations across representations nor a description that accounts for a full coordination among representations. We claim that there are several ways of coordinating these transformations across representations.

literature, studies about transformations of functions have been made using this traditional approach (Eisenberg and Dreyfus, 1991; Goldenberg et Al, 1988; Goldenberg & Kliman, 1990) showing the difficulties of students in dealing with visualization in transformation of functions and scaling of graphs.

In this study, a model that inverted the order of the traditional model was developed. The model starts with visualization of graphs, then focuses on the relationship between graphs and tabular values and, finally, focuses on the relationship between graphs and algebraic representations. We believe that the emphasis on visualization, besides restoring the historic emphasis on visual aspects in the study of transformations, allows students to easily move into the algebra and have some visual meaning for the algebra developed. Of course, this model is facilitated by the features of FP, as described in the last section.

Based on this model, a sequence of tasks was developed. Ron was introduced to the features of FP in three tutorials dispersed throughout the teaching experiment. In the tutorials, cubic functions were used; in the main part of the experiment, three other functions were used: absolute value, quadratic and step functions. The major part of the experiment consisted of three main problems. In the first one Ron had to match different pairs of graphs using the graph window icons. This task emphasized visualization and mouse actions as a way of investigating transformations of functions. Secondly, Ron was asked to transform the graph of $y=|x|$ in various ways and predict the effect of it on tabular values. After he finished his investigations with the family of absolute value functions he was asked to try his findings on quadratic and step functions. The transformations icons were thought as playing a fundamental role in this task as a way of transforming the functions, since the students have not been introduced to the algebraic facilities of FP yet, even though Ron ended up only using them after being asked to do so by the interviewer. Finally the students were asked to study the relationship between changes in different graphs and changes in coefficients of their algebraic equations. After these three exploratory problems, four more problems were given in an attempt of "tying up" the teaching experiment. In the next section, four "episodes[6]

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[6] An "episode" is an excerpt of the case study that forms a significant part of the story we seek to describe in this paper. For a fuller view of the case study, see Borba (1993).

will be discussed. As needed the specific problem that Ron was working on will be described.

3. Ron's Experience: Discrepancies and Coordination Among Representations

3.1 The Reference Parabola

Ron went really fast by the two first exploratory problems (matching pairs of graphs and studying the relationship between changes in the graph and changes in tabular values). This episode illustrates how students dealing with open-ended problems, an "open" software, with an interviewer[7] who would be most of the time challenging him and having the time to think can develop original strategies and help teachers to understand the importance of inquiry and the potential of students.

Ron was at this point working on the following exercise:

Problem 7: Transformations and coefficients
How would you investigate the relationship between coefficients in the equation of a function and the graph of the function?"

If the subjects did not come up with any idea three options would be given to them:

a) to perform an action on the graph and study the effect in the equation;
b) to perform changes in the coefficients of the equation and study the "transformational" effect on the graph;
c) to start as in either a) or b) and use the table as an intermediate step.

No matter which option the subjects chose, they were repeatedly asked to make a prediction before they used the software, since the "hide transformations command" was supposed to be off.

Ron had decided to start his investigations on this problem using a quadratic function written in the form[8] $y=ax^2+bx+c$. After a first moment in which Ron opted to study the effect in the coefficients of changes in the graph made through

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[7] The interviewer was the first author of this paper.

[8] The design of the study predicted the use of the equations written in the form $y=Af(Bx+C)+D$ since the equations in use could all be written in this form and this form facilitates the identification of transformations. In particular, this form can be seen as separating in the algebraic representation the effects of horizontal and vertical actions on the graph. However, since the format of the interview is open and Ron chose to start using another format, the interviewer allowed him to pursue his own inquiry.

the icons, he inverted the direction of his investigation. He was now studying the results in the graph of changes in the coefficients.

Using this new strategy, Ron tried several values between zero and 10 for "b" in $y=x^2+bx+5$. He produced the picture shown in figure 3 and conjectured that the vertices of this family of parabolas were generating a new parabola: "it makes

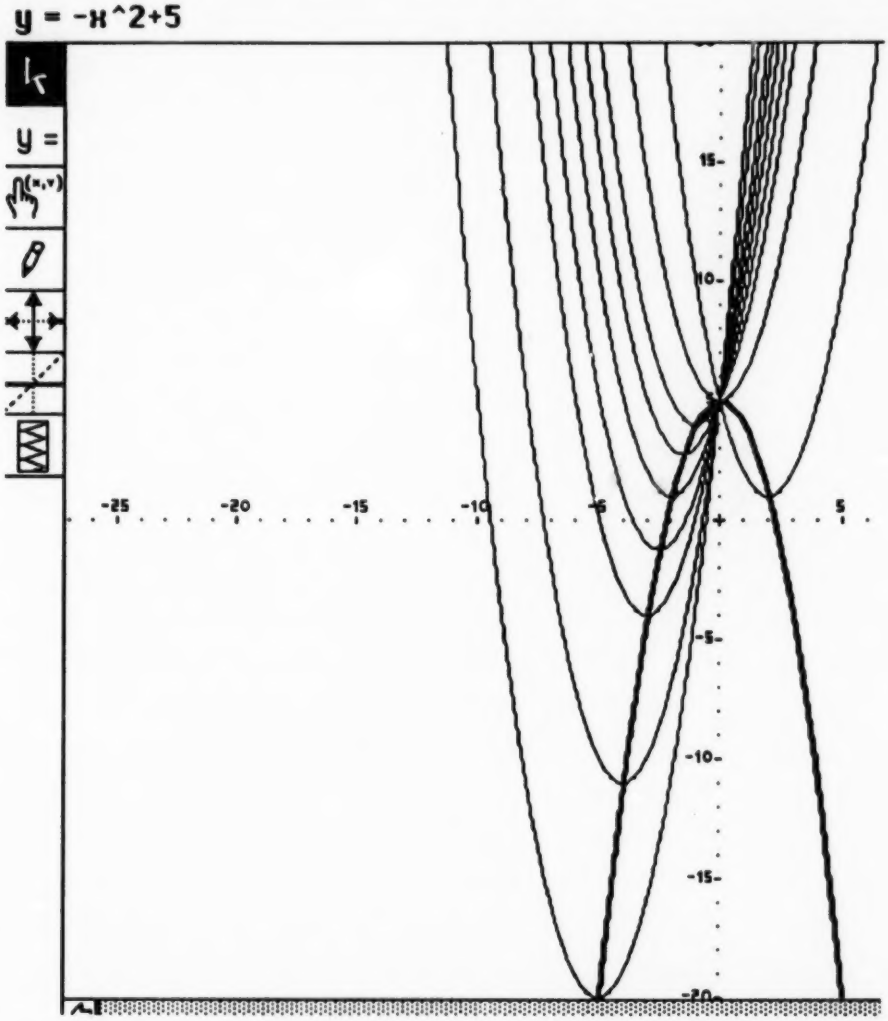

Figure 3- The family of parabolas developed by Ron in which b varies while "a" and c remain fixed. Highlighted is the graph which Ron eventually labeled the "reference parabola."

another parabola upside down this way where the vertices are" [4:5] [9]. He tried b=-4 to assure that the same pattern would happen on the positive part of the x-axis as well.

The interviewer asked Ron what he could infer from what he had found. He concluded "that the middle term is not just one transformation. With the coefficient of the middle term does not define just one transformation" [4:6]. It should be noticed that Ron's reasoning - linking a change in b with *two* transformations - suggests that he had been influenced by the design of FP which allows for only horizontal or vertical transformations without considering "diagonal" transformations. In this case, for instance, he was seeing that the graph had not been stretched and he knew that a horizontal and a vertical (more than one, therefore) transformation (maybe a translation) took place. Although he never labeled the transformation that was taking place, he was able to figure out that the parabola that crossed all the vertices of the family of parabolas $y=x^2+bx+5$ was the "reflection [around y=5] of this one [$y=x^2+5$]" [4:6] as shown in figure 3.

For Ron seeing the shape of the reference parabola had convinced him it was a parabola. He got its equation by observing it as a vertical reflection around its vertex. Another (algebraic) way of seeing that the curve that goes through the vertices of a family of parabolas $y=ax^2+bx+c$, where "a" and "c" are fixed and b varies, is a parabola is to notice that the set of vertices of the family of parabolas has the general expression $(-b/2a, (-b^2+4ac)/4a)$. As "b" varies, the vertices will vary as a parabola since the x-coordinate varies linearly in relation to b, while the y-coordinate varies quadratically. If one thinks of the above expression as a parametric equation, then $x=-b/2a$ and $y=(-b^2+4ac)/4a$ and y is a quadratic function of x. In the case Ron was working with a=1 and c=5, therefore $x=-b/2$ and $y=-b^2/4+5$, and from the first formula b=-2x. If this result is plugged into the last formula, the result is $y=-x^2+5$, which is the result Ron found.

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[9] Ron started calling the parabola that crossed the vertices of a family of parabolas - generated by varying only "b" in $ax^2+bx=c$ - the reference parabola.

The notation [4:5] means that that the excerpt was extracted from day 4 of the teaching experiment and is located on page 5 of the transcripts.

The notion of the reference parabola was left aside for a short period of time as Ron became involved with another problematic (Confrey, in press), even though this notion would resurface. Ron came up with a set of formulas to answer the general question of problem 7. His algebraic approach will be described in the next item. Before his algebraic approach is described it should be said that even though he had figured a solution, he still felt the need to understand what each one of the coefficients "a","b" and "c" in $y=ax^2+bx+c$ meant. He had little trouble with "a" and "c" but for "b" he had only the reference parabola with which to work.

This how Ron explained the meaning of the coefficients of $y=ax^2+bx+c$:

'a' is like . . how fast it goes upward . . . like zero is like this . . and go like this . . [he showed with his hand the curve moving from a horizontal position to a position that approaches the vertical line of the computer screen and the camera] . . so that's what a is . . , you make a negative and the whole thing goes downward. . .It's kind of a scaling, a stretch by this [shows with his hand a rubber sheet being HSed[10]] . . you know that if you do a HS or even a VS , you know that 'a' is gonna change . . . . but it's not going to change if you do a . . . translation, because the whole thing shifts, it doesn't change what the scale is, and a is a scale it's not a shift, . . b is the position on that curve . . .it takes your function . . . . I am not sure how to describe b . . [5:7d]

Ron seemed to be fairly comfortable with "a" as a scale and described it in relation to the terminology that had been used in the interviews (HS, VS, . . .). Later he will explain what b and c meant in a more visual way, but one which was disconnected from the "new transformations formulas" he had developed (to be discussed in the next item).

Ron explained b using the concept of the "reference parabola", which was $y=-x^2+5$ in this case. He explained that changes in b make a parabola move across the reference parabola as shown in figure 3, and that "the c value will shift this whole thing [shows a hypothetical parabola moving upward with his hands]..." [5:7e]. He also suggested later that changes in "c" would mean a new set of parabolas with a new reference parabola as well. In this case he was also using the reference parabola to describe the role of "c".

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[10] In this paper, VT stands for Vertical Translation, VS for Vertical Stretch, HT for horizontal translation and HS for horizontal stretch. Also HSed stands for horizontally translated and so on.

Ron's creation of the reference parabola seems to show the importance of an open inquiry, combined with and open design of FP and the fast response that computer based exploration can provide. The reference parabola allowed Ron to give "graphical meaning" to a varying parameter. With the reference parabola, Ron seemed to have found a language to describe the coefficients of this format of quadratic equation analogously to how slope and y-intercept were used by him to describe the meaning of "a" and "b" in y=ax+b.

3.2. Ron's Algebra

As mentioned in the last item, Ron came up with a very general rule for the transformations with which he was working. To understand Ron's solution for problem 7 it is necessary to go back to some of his reasoning as he was dealing with problem 5 (Problem 6 was just a tutorial focusing on the algebra facilities of FP).

The statement of problem 5 was:

Given the graph on the screen and two columns in the table window that store coordinates of points of this graph, what happens to the points that were in the table when you translate the original graph. You can use the *sampling command* or others after you have made your prediction." (*The "graph on the screen" was* $y=|x|$. *After Ron answered this question he was asked to answer the same question replacing translations by reflections and stretches*).

Problem 5 was not very challenging for Ron, especially the first part regarding translations, which he went through both without any hesitation and without using FP. Although reflections were not that problematic either, his solution showed some of his algebraic approach which is embedded in geometrical meaning.

Ron argued that a rotation around the x-axis would make every y-coordinate be multiplied by -1, because "anything on top is going to be reflected to the bottom over . . which is in the negative range" [3:10]. Next, Ron choose to describe how he would vertically reflect through a reflection line of the form y=b.

Let's say the reflection line is at 10. . . Then at this point at 0 would reflect up to 20. A point at 10 would stay right where it is. A point at 5 would reflect up to 15. And

you can . . . I guess the formula for that would be you take the y-value that you have, you subtract 10, multiply by negative 1, and then you add your 10 back in. [3:11]

On the first part of this excerpt, Ron was using what seemed to be a visual model: a point at 5 would reflect to 15, and one at the reflection line would remain where it was. Then Ron suddenly shifted to a different kind of reasoning, which also had roots in visual thinking: he argued that 10 would be subtracted from every point, the result multiplied by -1 and finally 10 added to that result.

Even more impressive was Ron's ability to code in co-variational equations[11] what he had just said. He wrote down the formula (y[-10])•-1[+10] [day 3: scratch papers], which could be interpreted as y'=(y-10)*(-1) +10, where y' is the new reflected y-coordinate with the anchor line being placed at +10. As Ron was writing the above formula, he made clear the "generality" of his rule. He said, as he responded to the interviewer's probes, that "y minus 10 times negative one plus 10 . . . the subtract 10 brings the line down . . . Then the -1 flips it over the x axis and the +10 moves it all back up where it should be" [3:11].

Ron seemed to have at least one visual model, which could be supported by the algebra developed on his scratch paper. He was so confident in this model that he saw no need to try it in FP. Only upon the insistence of the interviewer did Ron test his idea using the table window to generate points and send them back to the graph window. Thus, he "re-assured" himself that his model worked. This kind of algebraic reasoning was so strong in Ron's reasoning at this point that he also asked about the algebra facilities of the table window (which according to the design of the study would not have been introduced until the next tutorial, problem 6).

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[11] We use the term *co-variational equations* to refer to the equations that different students who were interviwed, with their different styles, used to express changes in either the y- or x-coordinates when a graph was vertically or horizontally transformed. This terminology is heavily influenced by the work of Rizzuti (1991) who used *co-variational view of function* to refer to a view of functions that is largely structured by tables of values. The terminology *co-variational equation* was first coined by Borba (1993).

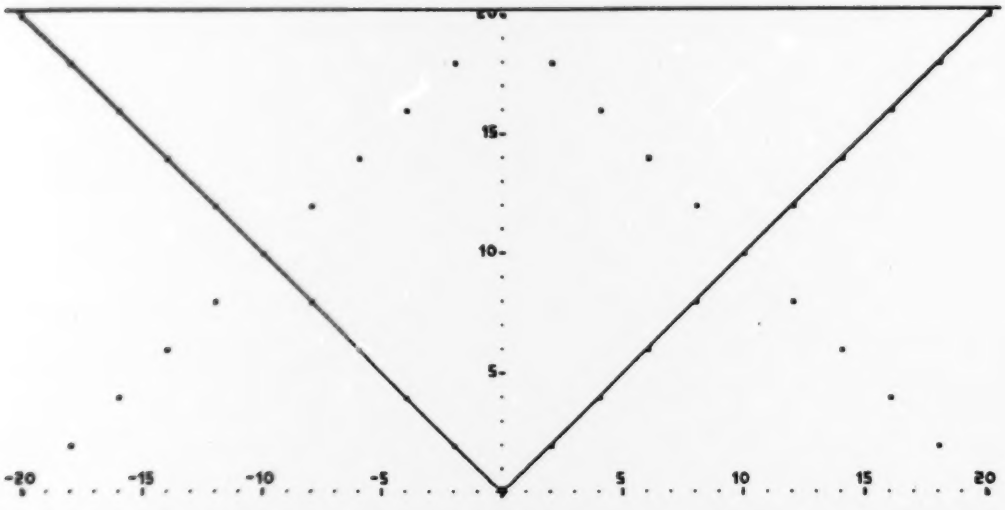

Figure 4 - y=abs(x), sampled points on this graph and the reflection of these point across the line y=10. The horizontal line on top is the top of the computer screen Ron was working on.

After Ron sent the points to the graph window (see figure 4) the interviewer, playing devil's advocate once more, asked him why the vertex had "shifted" 20 units instead of 10 as his model "suggested." The question did not confuse Ron at all: "That's because when you bring this [the graph of y=|x|] down it [the vertex] goes to -10 not zero... This slides down here [-10], you reflect it so it goes up to 10 you add 10 so it goes up to 20." [3:14]. Ron seemed to be expressing a "perfect" integration between visual laws and algebra, and it seems that he would have been able to extend this harmony even to the table, since he used the algebra to generalize (from the beginning) the more "empirical" results that other subjects had to get through before starting to "algebratize" their findings.

Soon Ron generalized his previous equation replacing 10 by "T". He wrote (y[-T]).-1[+T] [day 3: scratch papers], which could be interpreted as meaning y'=(y-T)*(-1) +T. The interviewer tried other probing questions regarding different reflection lines, but Ron had no problem in applying his model to different values of the reflection line.

As Ron started to investigate stretches, he claimed that "it's basically the same equation [the one described in the paragraph above]" [3:17]. He explained that it was the same since one can think of the anchor line in the same way that he had

previously thought of the reflection line. He also had no problem saying that multiplication was the operation associated with stretching, even though he could not come up with an argument for why would that be so. Also, to connect both stretches and reflections he argued that "if you have a stretch by -1, that it's the same as a reflection" [3:18], which seemed to have made him even more confident that the formulas should not differ.

For a long period of time, this general equation shaped Ron's answers to all the challenges presented by the interviewer, since he would just refer to the formula as a way of answering the question. For instance, when he was explaining what happened to the point (5, 5) of $y=|x|$ in a HS (s=3, a=5), he would say: "you subtract 5, which is where the anchor line is, from all the values. . .from all your x-values, you multiply your x-value by 3, and then add 5 to move it back over here." [3:19]. The interviewer wrote down the expression (5-5)3+5" [day 3: notes] as the operations that would be done to the x-coordinate 5, and asked Ron whether that was what he had in mind. Ron agreed and later on showed that his model x'=(x-a)s+a (where s is the stretch factor, a the position of the anchor line) would work[12].

The algebraic approach Ron used in the table window, as it was summarized in the last paragraph, resurfaced as he was working on problem 7 (the one that proposes a study of the relationship between changes in coefficients and changes in graphs). Previously, Ron has found, using the icons and the rescale facilities of FP that a HT by 5 to the right would imply in a change in "c" from 5 to 15 in the equation $y=x^2+3x+5$. Despite his finding, he did not seem to be very sure of it and wanted to check it, using paper and pencil and his previous results from the analysis of co-variational equations. He claimed: " . . .*I know an easy way to find out what c will be . . . Just substitute* . . . do the formula that I did [the ones he used in problem 5]. y. . .so x, x^2 + . . . I think this will work . . . 3x + 5, and the thing . . . transformation was to the right by . . . It's $(x+5)^2$ +3(x+5)+5." [4:18]. He wrote down what he thought was going to be an algebraic formula for a HT by 5 to the right: " (x, x^2+3x+5) --> $(x+5)^2$+3(x+5)+5 " [day 4: scratch paper]. Next he developed the above expression until he reached "(x+5, x^2+13x+45)" [day 4: scratch paper]. Ron had then a discrepancy in the results that with which he had to deal: he had found using the

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[12] Ron did not write the equation in this manner. This is our interpretation of his argument that involved oral explanation and notes on the scratch paper.

icons that a HT by 5 to the right would make "c" change to 15 and now, using his algebraic algorithm he found c=45.

Until this point the "hide transformations" feature of FP had been on. The interviewer now turned it off and FP displayed y=(x-5)^2+3(x-5)+5. Ron's first reaction indicated that he did not expect these results: "I went the wrong way" [4:20]. It seems that he could come up with a "visual explanation" of what happened, but he also had to find a way of making sense of "x+5." This tension illustrates how co-variational equations can conflict with the algebraic form built into the software for displaying transformations in the graph window. For instance, in the table one. might look at a HT by 5 as being expressed by the expression x'=x+5, where x is an original coordinate of a function and x' the transformed one. This transformation is expressed in the y=f(x) form as y=f(x-5).

The tension discussed above left Ron with a dilemma to solve: FP displayed a graphical result that he was convinced was right (probably due to the "authority of the computer"), but he could not find an explanation with which he was satisfied. He criticized FP for its lack of flexibility, arguing that since x is the variable that is changing in a HT, he should be able to make changes in the left hand side of the equation displayed by FP: "if this could plot x+5 instead of x, it's plotting all these functions onto x, because that's how a function works . . .I want to plot a function on x+5" [4:23]. It seems that he thought that if he could plot f(x+5) he would solve the problem, but this idea did not help him to understand the FP display. The tension Ron experienced illustrates how the discrepancy among representations and the constraints of the system can produce instability and potentially lead to confusion or further investigations.

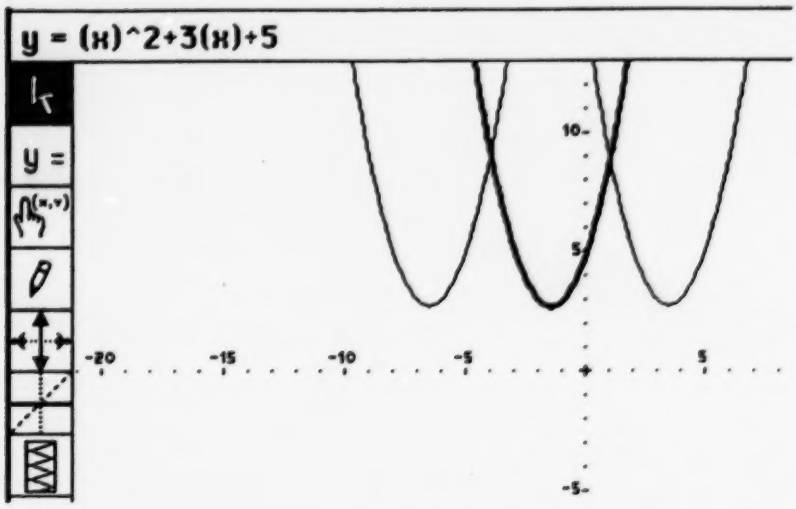

Figure 5 - From left to right, the graphs of $y=(x+5)^2+3(x+5)+5$, $y=x^2+3x+5$ and $y=(x-5)^2+3(x-5)+5$.

In any case, Ron was convinced that his algebra calculation- which led him to find c as 45 - was correct. Soon after, Ron could link his previous visual understanding of the HT with the algebra display of FP. Pointing to the graphs of $y=x^2+3x+5$ and $y=(x-5)^2+3(x-5)+5$ (see figure 5), he argued that the latter function would have 5 subtracted from its x-coordinates so that it could have the same y-value as the former function. However, as he was asked by the interviewer whether he was "happy with this explanation" [4:26], Ron answered that " my logic isn't finished . . . because this is x+5" [4:26]. Ron complained again about the design of FP:

x is usually considered to be the first coordinate. . . . But here, x is not the first coordinate because the first coordinate is x+5. . . . .The question is when do you find x? Do you take x. . . does it take x before you do the transformation or after you do the transformation. It's obviously after you do the transformation [on FP]. [4:28]

In addition to his complaints about FP, he concluded that FP takes x after the transformation which might be interpreted as: FP cannot operate with "x+5." In any case, it seemed that he was trying to coordinate representations.

After some effort, Ron reached a satisfying understanding for himself of what the FP algebra display meant and then he tried to connect the algebraic "micro-world" and

the visual one. After a sequence of attempts, he was able to coordinate his description of the HT by 5 as "(x+5, x^2+3x+5)" with the FP display that showed "y=(x-5)^2+3(x-5)+5. He came to the conclusion that he could re-write his model as (x, f(x-5)) and (x, (x-5)^2+3(x-5)+5) because[13]

"It's like this. . .you translate by 5 and you get (x+5, f(x)). . . . . Then you subtract 5 [from each side] x+5-5, so it's x-5 . . will give you (x, f(x-5) there. . . . . [because] you want x to be the same, so you do (x, f(x-5)). [4, 37][14]

Note that Ron's reasoning can be justified if one thinks of his expressions as parametric equations. In this sense, (x+5, x^2+3x+5) can be changed into the form y=f(x) if we make x'=x+5 in a HT by +5 (the y values do not change) and take into account that y=x^2+3x+5. From the first equation we derive that x=x'-5 and, substituting in the second equation, we have y=(x'-5)^2+3(x'-5)+5 which is equivalent to (x', f(x'-5)) as proposed by Ron, or y=f(x').

After some checking by the interviewer, to assure that Ron was convinced of what he has just said, the interviewer decided to check if Ron could really put together the graph and the different algebra representations: "What I want to know now is when you look at these two here (the two graphs on the right of figure 5) . . . How on earth it's not x+5 and it is what the computer shows?" [4:40].

Ron's response shows his confidence:

R: If you take a point here of x, right? . . . That gives you this point here on the graph. So take your point here of x, it gives you a point way up here [pointing at the original graph, which is the middle graph of figure 5] , okay?
. . .

R: But you're moving over by 5, so you want the one that's over here. So if you're here [pointing at vertex of the HTed graph] of the..., right? Normally it would be up

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[13] The reader should take note that the reasoning involved in the last paragraph is going to be fundamental for the last episode to be discussed on this paper.

[14] Ron was both thinking aloud and writing. In the quoted passage, his utterances are represented using algebraic expressions that he actually wrote on scratch paper on day 4.

here. . . *If I were here on the first graph, it would be way up here. You don't want way up there. You want what's over here.* So you say. . .

. . .

R: What you want is transformation by 5, so you want what's over here, you don't want something way up there ... So, you just take a sample of what's over here, *so you move back 5* and see what's there and that's what you put down instead of putting that. [4:40-41]

Ron seemed to be able to coordinate "the move back" by 5 (a subtraction by 5) with the algebraic language "x-5" used by FP. Also before he was explaining that the image of a point such as 3 would be "way up there" for the original graph but in the new graph "you want it over here" and if "you move back" by 5 you can get the same "low" image of the original graph (see graphs on figure 5). Ron had no problem extending his coordination of representations to the table window. It seems that he had coordinated the table algebra and the graph algebra if one looks at how he changed "x" into "x+5" and then applied it to the equation $y=x^2+3x+5$ (see figure 6).

x	y	m=x+5	n=(m-5)^2+3(m-5)+5
-5.0	15.0	0.00	15.00
-4.6	12.4	0.40	12.36
-4.2	10.0	0.80	10.04
-3.8	8.0	1.20	8.04
-3.4	6.4	1.60	6.36
-3.0	5.0	2.00	5.00
-2.6	4.0	2.40	3.96
-2.2	3.2	2.80	3.24
-1.8	2.8	3.20	2.84
-1.4	2.8	3.60	2.76
-1.0	3.0	4.00	3.00

Figure 6 - Ron's table: making sense of "different algebras."

As he gained more confidence he also regained control of the investigations and without being asked said he knew what to do next. He wrote down on the scratch

paper the development for $y=(x-5)^2+3(x-5)+5$ in the following way: $x^2-10x+25+3x-15+5$. He then compared this equation with the original equation $y=x^2+3x+5$. From this, he created a general strategy by which he could transform any equation of the form $y=ax^2+bx+c$ into its new $y=a'x^2+b'x+c'$, when the former was HTed by T.

new $c = c - T \bullet b + T^2 \bullet a$
new $b = b - T \bullet 2$
new $a = a$ [day 4:scratch paper] (rule for coefficients)

Ron was able to generalize a change in the coefficients. This generalization was connected to the experience he had just had in his attempt to coordinate co-variational equations with algebra of the form y=f(x) ("graph algebra"). There is no indication, however, that this generalization was connected to the visual argument he developed earlier when he explained why a negative sign in "x-a" would imply in a shift of the graph by "a" to the right. However, it suggests that Ron, who had twice called himself an "algebra person" during this interview, felt more comfortable when he was able to rely mainly on algebra manipulation.

This episode illustrates how Ron's frustration with FP was turned into a new problem by the interaction between him and the interviewer. It also illustrates how discrepancies among representations can become an "epistemological lever" (Borba, 1993) until he could use one representation to justify a pattern met in the other one, as in the case of his visual justification to coordinate the co-variational equations and the graph algebra. Finally, Ron showed that when he could see a way of solving a problem by means of algebra, isolated from other ideas, he had no problem carrying out the algebra, as illustrated by his "rule for coefficients", a very powerful rule to model transformations in parabolas written in the form $y=ax^2+bx+c$. In the next episode we will describe how Ron comes up with a new coordination between transformations on the graph (driven by actions in the graph) and changes in the coefficients.

3.3. Ron's Coordination of Actions on the Graph and Algebra

As Ron has achieved enough responses with the quadratic equation written in the form $y=ax^2+bx+c$, the interviewer asked Ron if he could quantify the meaning he had for "a", "b", and "c". In other words, the interviewer asked him if he could say something such as "if I change 'a' by a multiplicative 2, this means a stretch by 'n'."

Ron became somewhat discouraged and the interviewer suggested that he investigated the issue of quantification using the format originally designed for this teaching experiment: $y=Af(Bx+C)+D$. Ron accepted the suggestion, even though he had to also attach some meaning as well to these new coefficients (A, B, C, D). After a period of time in which Ron was search for sense for these new coefficients and a "quantitative meaning" as well for them, Ron started to build a quite original model to coordinate the graphical and the algebraic representation. After some trouble with understanding what C meant, Ron decided to keep A=B=1, D=0 and vary C to produce $y=(x+C)^2$:

Wait, wait, I just want to change it back to $+5^2$ $[(x+5)^2]$. That's all. So now *if you imagine this x-axis shifting over completely by 5.* Shifted +5, okay? . . . Now, and change all the values back to what they should be okay? And then over here at -5 is going to be where the y-axis is going to be -5 is going to be, see what I'm trying to say? [6:16]

Ron was building an argument in which he was seeing a HT as a "function" of moving the x-axis instead of seeing it as a shift of the graph. As the interviewer realized the nature of Ron's argument, he asked Ron if that could explain why, when C had a "+5 change", the graph would move to the left. Ron's response was quite telling: "Um. . .yes. If you do . . . if you think about it right. Let me think about it. Yeah. See if you pull this to the right, -5 is getting to end up at the zero, and in the distance will be 0 at -5, see? [6:17]. Ron's articulation suggests that he was understanding "+5" as a movement of the x-axis to the right. The x-axis would be pulled back to the right because "[b]y *being a function, it pushes it all back the way it was supposed to be* . . . . so then this whole thing slides back and then it's plotted where it is now . . " [6:18]. His explanation can be understood if one considers that by function he meant f(x) instead of f(x+5). If this were the case, it would mean that he found a way to get around the limitation in the syntax of FP algebra display that he had complained about. In any case, if that was what he meant by "function" then it he had to pull back the axis so that the origin of the Cartesian plane would remain (0, 0) instead of (-5, 0) which it would be after the axis had been pulled the first time. According to this interpretation, as the axis was pulled back, the graph moved to the left. It should be also noticed that this way of seeing the translations represents the specific way that Ron was "shaping" the graph representation to fit his way of thinking. FP does not have any feature in its design to suggest this kind of interpretation, nor were the tasks designed with this goal. Nevertheless, as the

reader will see, Ron used this interpretation to coordinate different aspects of his knowledge .

Next, he tried the same approach to B: "This 2x [in y=(2x+5)] is like taking this whole thing [x-axis] and stretching it along this way, just like we pulled it over this way, and now *we're stretching it* . . . the same line." [6:19]. The interviewer asked Ron why the graph would be contracting - in comparison to y=(x+5) - if the axis were being stretched. Ron and the interviewer took a fair amount of time to come to the agreement that what took place would be a *contraction* of the axis. However, the interviewer influenced Ron into accepting an idea that was not consistent with his model. In Ron's model the axis should be stretched, the function graphed and then the axis should be released "because it is a function." This release would generate a contraction for the graph. Neither Ron nor the interviewer seemed to realize that they were both working with a new model. This new model consisted of a pure contraction of the axis so that the graph would look contracted in comparison to the original graph. In contrast, Ron's original model consisted of stretching the axis, then releasing the axis resulting in a contraction of the graph.

A long time was spent clearing up this issue. As it became more clear to the interviewer what Ron's original model was, the assisted Ron in keeping the model stable. Ron's original model stated - as it became clear during the interview - that the change in a coefficient "inside the function" would be a transformation of the x-axis and not on the function. In this sense, he would stretch the axis by 2 when B changed from 1 to 2 or he would move the axis 5 units to the right when C changed from zero to 5. Next, the graph of the prototypic parabola would be "plotted" and the graph would "bounce back." It is important to notice that this model makes an "non-inverted" relationship between the change in the coefficients and the action that is made on the graph. In contrast, the interviewer's model, based on the FP design, did not have such a correspondence: a change from 1 to 2 in B meant a HS of 1/2 and a change in C from zero to 5 meant a HT of 5 to the left (or -5).

A complete settlement of this investigation did not come until later. For now Ron shifted his investigations to the A coefficient. Ron changed A in $y=x^2$ to 3 making $y=3(x)^2$. He predicted that the graph would be "expanded." He graphed the new function using FP, saw that the graph had actually contracted, could not explain the discrepancy and decided to try to change D instead. He changed D from zero to 20

and got the equation $y=(x)^2+20$. He was completely amazed that the graph was not in the grid visible on the screen (the range for the y-axis was [-19.7, 19.7]): "Where'd it go?" [6:25].

What amazed the interviewer was not that Ron made a prediction that did not make sense (for the interviewer). The interviewer seemed to be quite surprised that, after several probing questions, Ron still could not come up with any explanation of "what happened" to the graph. It is interesting to notice that in spite of Ron's skill at manipulating algebraic equations, designing general equations that described translations, reflections and stretches and coming up with visual models such as the one about stretching the x-axis, he still had difficulties connecting algebraic and graphic representations with D, which from the point of view of the interviewer was easier.

Ron now had two open issues: A and D. Abandoning his initial plan to finish one coefficient before moving on to another, Ron moved to another exploration. He tried, for a while, to extend his previous reasoning of shifting the x-axis to one of "squaring" the axis. In other words, he was thinking of linking the squaring in the algebra to a squaring of the x-axis. This idea, although creative, did not lead him to new insights, other than the fact that, after he dropped this idea, he was able to "re-elaborate" his original model regarding stretching and translating the x-axis:

And then you have a function. This pencil, okay? This is y=x right here. Plop. And if you don't do anything, it just plots right on the paper, it sticks there, and that's what you get. Now when you say y=x+5, shifts the graph over, plops it down, or shifts the graph over, plops it down and moves it back where it's supposed to be, okay? So now the functions move to the left just like that .... And no, so if you do y=2x, then it says, take the whole graph paper and stretch it out, plop the x down, plop this - would that stretched piece of graph, will that stretched piece of paper, put this on there, then push it back to where it's supposed to be and this whole thing will go like this. [6:32]

At this time, the interviewer could see what Ron had in mind and asked him if the "double rubber sheet model"[15] would help him. Without hesitation, Ron incorporated the rubber sheet metaphor in his discourse:

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[15] The double rubber sheet model was proposed in Borba (1993) inspired by the work of Goldenberg et al. (1988) of the rubber sheet. The double rubber sheet model proposes that we think about the

So, when you change B and C, it's like taking the graph paper and stretching it and putting it back to normal, or you change A and D, it's like taking the one in the front and stretching it and putting it back normal, or doing whatever. Stretching it and moving it, do you know what I mean? [6:35]

From this quote it can be inferred that Ron was seeing the double rubber sheet metaphor as a solution for the overall problem: he would use the back rubber sheet, where the axes were, to transform the x-axis "before the function" was applied, a transformation that can be related to B and C. Then, the front sheet would be used to make the transformations connected to A and D. But the interviewer seemed to notice something in Ron's hand movements that indicated something different:

I: Well, this graph here is only doing this or this [HSing] or moving this or this [HTing], *. . . because you can also think of a stretch like this, right? Or of moving this - moving down here, moving down here.*

R: *Can. . .using this formula, can you show me one that doesn't touch the x-axis?*[6:35]

The interviewer saw that Ron made no vertical movements with his hands. Ron's response gave a picture of how he was understanding the formula $y=A(Bx+C)^2+D$, and also provided a partial explanation of why he was surprised earlier in this episode when a change in D from zero to 20 units took the transformed graph out of sight. According to his hand movements (very imprecise), a change in D would HT the graph to -20, similar to the way C would. His understanding at this point again illustrates how his algebraic knowledge, evidenced throughout this teaching experiment, did not necessarily help him understand what role D played in the above formula, or he is thinking of something else that these authors can not model.

As the interviewer showed Ron that a change in D from zero to 1 would VT the graph of the prototypic quadratic function by one unit, Ron adapted his previous ideas. He argued at this point that "B and C. . . Anything inside the function, stretches and pulls and shrinks it this way or moves it this way [horizontally].

Cartesian plane as two superimposed rubber sheets. The one in the back with the Cartesian axes on it and the front one where the graph is printed. Ron is going to develop this model, adding strings and an observer that see things "coming back".

Anything outside the function stretches or pulls it this way [vertically], so now only one rubber sheet changes . . . " [6:36].

For now Ron did everything on the back rubber sheet (with the axes on it) and used the front transparent rubber sheet just to plot the results. Moreover, he had an explanation of why different coefficients would be connected to the x- or y-axis:

Because b and c are inside the function. That means you haven't done the function yet, so you're still working on the x-axis. When you're outside the function, you've already done [with] the function and you're working on the y-axis now - directly with the y-axis. [6:37-38]

Next, Ron investigated what would be the amount "n" of a VS when a change "m" in A was made on the graph of $y=x^2$. He soon realized that if A were changed from 1 to 2 the graph would be VSed by 2. He had no problems with D either, and the interviewer asked him if he now knew what happened to the graph of $y=x^2$ when D was changed to 20, as earlier in the interview. Ron had no problem this time saying that "it's up there." [6:41]. Trying to raise new conceptual issues, the interviewer asked him "why does a change by 10 [in C] move the graph in the negative direction, while a change by 20 on D shoots it up and not down?" [6:41]

Ron and the interviewer had a sequence of interactions in which clearly both were "subjects" and each was trying to learn what the other saw. As the following excerpt shows, Ron and the interviewer were starting to come to an agreement about the best way to use Ron's model:

R: Okay, you have two rubber sheets, *I have a rubber sheet and a thread,* okay?

I: Okay.

R: *The thread is the graph which is in the shape of a parabola. B and C, you change B and C so you stretch your rubber sheet* . . . it's a sticky rubber sheet, okay?

I: Okay, yeah.

R: So you stretch your rubber sheet this way or this way. You push the sticky . . . push the string onto the rubber sheet, then you let it go back to normal.

I: Okay, good.

R: Now, whatever you do to this, the string automatically goes with it, okay? So now when I change D, the string automatically goes with it. Because the string is already pressed on there now.

I: Okay, then when I think . . . but then, how is that going to move [vertically]? It's already there.

R: So if I shift it up 5, it just goes up 5.

I: So it's not stiff there, it can slide? [referring to whether the graph translates with the x-axis at the same time or whether the axis stays fixed]

R: Yes. No, the whole thing is moving up 5.

I: Well, if the whole thing moves up 5, the graph's not going to go up 5.

R: It just did go up 5.

I: No, it's just going to be like if I cut here and put it up here.

R: Wait. Okay, I see what you're saying. Maybe a row or two rubber sheets would work better.

I: So that's it . . . the first step, I like it. You got there, then how do you go from there. When I add 5. . .

R: I was thinking the. . .what's. . .after you did the first part, *then you already have the equation, you already have the function on your paper, and then whatever you do the function still goes with it, just like it does here, so when you add 5 to D, it just goes up 5.*

I: Hmm hmm.

R: But you're saying that it might work better if you think of it as having the rubber sheet move too, I mean, as having the function move after.

I: Yeah, well, when I see this here, if I think of 2 rubber sheets, I'm getting the clear one that was stuck in the bottom pulled in at 5. . .

R: Right. . .

I: My next . . .

R: Okay, that's better. That'll work.

I: Okay, you know, or if I want a . . . but I like because you have the tendency to think the opposite way that I do, so that's why I'm happy with it. Because you're thinking of doing everything in the axis, and I'm not doing anything in the axis. . . . [6:45-47]

This interaction suggests that Ron had a model of the rubber sheet and a thread that worked until vertical movements had to be done. Using the double rubber sheet, Ron made a distinction between "actions in the x-axis" (on the rubber sheet in the back) for the horizontal transformations, and "actions on the graph" (on the front transparent rubber sheet, where the thread was attached) for the vertical transformations. In this sense he also answered the interviewer's question regarding the discrepancy of VTs and HTs in Ron's former model, in which all the actions were made on the axis.

Ron and the interviewer kept "working" on the details of the model. "Pins" were brought into the model as a way to link rubber sheets after one of them was stretched. For instance, if the back rubber sheet were stretched by 2 (change in B from 1 to 2), then a transparent rubber sheet with a prototypic parabola on it would pinned down to it, and both rubber sheets would be released with the elasticity from the back parabola "commanding" the action[16]. Ron then used part of this argument to come up with his "overall generalization":

R: Here's what you do, okay? Here's the overall generalization. You have 2 rubber sheets, one of the them is your graph. The clear one is your ... the other one is your function.

I: Yeah, the one on the bottom is the axes.

R: Is the axes.

I: Okay...

R: So anything that is inside the parameters of the functions means you change the graph before you do the function. That means that you're . . . that means that B and C are a stretch and a contraction [he probably got confused and said contraction instead of translation] before you do your function, okay?

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[16] There are some operational problems to actually constructing such a model. But as a metaphor it proved to be very important for Ron and for the interviewer.

I: So you're going to do that on the graph or the axis?

R: On the graph. Or not on . . . on the axes, not on the function, not on the function.

. . .

R: . . . Then when everything is all stretched, you pin it down and you slap our function on top of it, then let everything bounce back the way it should be. That means that when you shift it to the right 5, plop it on, when you bounce it back, it will move it left 5, okay? If you look at your function, shift this over, plop it on, and move the whole thing . . .

. . .

R: Plop that on, move it over, okay? Now when you change A and D, what happens is you take your clear one, keep this where it is and you stretch your clear one up this way. Then, when your clear one's all stretched, you plop it on. [6:48-49]

The interviewer seemed to be satisfied with Ron's explanation and moved to other questions that did not raise more conceptual issues, such as: 'Can you undo a VS with a HS?' and 'What is the amount of change in B that is equivalent to change in A, in a parabola?' The interviewer also asked him what he could generalize from this experience with the parabola to the absolute and step function. Ron said that he did not see any reason that these other functions would work any differently.

In this episode, Ron showed how a person, with excellent algebraic skills, might not be able to understand what D in y=Af(Bx+C)+D represents. However, Ron "shaped" the graphical representation to something with which he felt comfortable, and was able to build a graphical/visual representation that has allowed him to predict transformations in the above formula. This episode illustrates how Ron acted upon FP, the interviewer's input, and the metaphors to construct his own metaphor. Ron moved from actions on the x-axis with "bouncing back", to actions on both axes, to the double rubber sheet metaphor, to the extended double rubber sheet metaphor and its accessories, to the final metaphor. The final metaphor linked the kind of changes that take place in the coefficients to the corresponding changes at the graphical level, consisting of actions on the graph and on the axes.

4. Discussion and Conclusions

In these episodes from a case study, we saw how one student created a path among a variety of representations in the company of an interviewer who followed his reasoning and pursued potentially interesting lines of questioning with him. We see such investigations as central to a constructivist research program, as we work to delineate the multiple methods one can use to make sense of the topic of transformations.

The case illustrates the point that visual reasoning, seeing graphical transformations as movements on or of the plane is a powerful form of cognition, and one which requires that we provide students adequate time, opportunities and resources to make constructions, investigations, conjectures and modifications. In times in which high speed computing makes the management of large amounts of data essential, we see such facility in visualization as increasingly important in mathematics education.

Furthermore, we would argue that the strength of Ron's investigation lay in his ability to coordinate those visual actions with changes in other representations. It seems that Ron was most novice in his visual explorations, and yet these were provocative to him. The reference parabola was his own discovery and motivated him to explore more deeply the impact of changes in "b" (as in $y=ax^2+bx+c$). As he sought to describe what happens in a reflection around a line that is not on the axis, he produced a verbal description. It was this verbal description of the visualization that he was able to code into algebra-- albeit an alternative algebra to that which is taught in school. We see the development of a covariational set of equations as a compelling mathematical tool. In many ways, one can argue that the covariational set of equations is a better complement to a graph and a table in that it corresponds so nicely to the transformations in the other two representations.

In our work (Confrey, 1992b) we have referred to the coordination and contrast among representations as an "epistemology of multiple representations." In this paper, we have illustrated that different representations provide a source of motivation to complete an inquiry or investigation.

In light of Ron's covariational equations, Confrey proposed a device for coordinating changes in graphical transformations with tabular and algebraic changes. She displayed the transformations as a series of modifications to the basic prototype. If a prototypic equation was y= sin(x) then the x's and y's in that relationship can be modified separately to produce a sequence such as the following:

y	=	sin (x)
|		|
(mult by 2)		(subtract 3)
y' = 2 y		x' = x-3
|		|
(subtract 3)		(mult by 3)
y" = y'-3		x"= 3x'

To describe y" in terms of x", one would have to reverse the changes from x" to x' to x then apply the function and move from y to y' to y". This gives one, y" = [2sin ((x"/ 3) + 3)] - 3. We have found this two dimensional display an effective curricular intervention with teachers. This illustrates how research on student conceptions can lead to curricular innovation.

After Ron had achieved a stable system in which he could describe the visual impact of a transformation and its subsequent impact on the co-variational equations, the interviewer provided Ron with the challenge of contrasting his algebraic form with that of the computer. More than any other episode, this one describes how using computers as tools results in forms of semiotic mediation (Wertsch, 1985). The computer software had a certain form of equation built into its design. This form conflicted with the algebraic model Ron had built. As Ron sought to resolve these two models, Ron created a new interpretation-- one which was: 1) consistent with his previous covariational algebra, 2) allowed him to make sense of the form of algebraic equation displayed on Function Probe, and 3) brought into this dialogical relationship (together with the interviewer) a visual metaphor of two rubber sheets that allowed him to visual resolution of the discrepancies. In some respects, as a result of this "intershaping relationship" (Borba, 1993) Ron developed his "own" software as he stretched and translated "axes", brought thread,

and tacks and glue into his metaphor and and argued for transforming Function Probe's algebraic display to allow for a more flexible treatment of variables.

This discussion gives a very "real" meaning to the claim that new forms of representation change the mathematics to be taught. Mathematics does not exist independently of its representational forms; it exists through those forms. Teaching mathematics with computers is thus not just a matter of placing mathematics onto a machine. Its structure, its processes and its claims for knowledge vary with its medium. The investigation we present here alerts us to the importance of revising our own understanding of mathematics in light of students' initiatives. Design must reflect student methods, curricular must evolve along with design and student methods will continue to change as a result of the resources available to them.

If classrooms are to avoid being left behind in such an evolution, they too must become a critical component of such a loop. Viewing the interview as a specialized form of social interaction, one in which the interviewer sought to understand and follow the method of the student most times, and to explore together with him at other times, suggests that the results of the interview studies can inform classroom interactions.

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